Transforms Part 2 & Image Processing

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ANY function $f(x)$ can be represented exactly as a sum of $\sin()$ functions with specific amplitudes and phases.
Fourier Representation
Homework

• Compute the continuous, infinite Fourier Transform of:

\[ f(x) = \begin{cases} 
0 < x < 1 : f(x) = 1.0 \\
otherwise : f(x) = 0 
\end{cases} \]

• Now say the function is finite, defined on the range 0<x<2. The transform is the same, of course, but the result is only defined for specific k values (see the lecture notes). Plot the sum of the first 10 elements of this Fourier series on the range 0<x<6. (turn in the plot) what do you observe ? What would be different if you used 100 elements ? What would be the same ?
Fourier Representation
Fourier Representation
Fourier Representation
Fourier Representation
Fun With Lasers II
Fourier Transforms

\[ \mathcal{F}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \]

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(k) e^{ikx} \, dk \]
Finite Fourier Transforms

\[ F_k = \int_{0}^{w} f(x) e^{-i(2\pi k/w)x} \, dx \]

\[ k \subseteq \text{integers} \]
Finite Discrete Fourier Transforms

Finite range in real space -> discrete in Fourier space
Finite range in Fourier space -> discrete in Real space
Finite range implies periodicity in the same space

\[
\bar{F}_k = \sum_{x=0}^{w} f(x) e^{-i(2\pi k/w)x}
\]
\[
\bar{F}_k = \sum_{x=0}^{w} f(x) e^{-i k x} \rightarrow \\
\begin{bmatrix}
\bar{F}_0 & \bar{F}_1 & \bar{F}_2 & \bar{F}_3 \\
\end{bmatrix} = \\
\begin{bmatrix}
e^{-i k_0 x_0} & e^{-i k_0 x_1} & e^{-i k_0 x_2} & e^{-i k_0 x_3} \\
e^{-i k_1 x_0} & e^{-i k_1 x_1} & e^{-i k_1 x_2} & e^{-i k_1 x_3} \\
e^{-i k_2 x_0} & e^{-i k_2 x_1} & e^{-i k_2 x_2} & e^{-i k_2 x_3} \\
e^{-i k_3 x_0} & e^{-i k_3 x_1} & e^{-i k_3 x_2} & e^{-i k_3 x_3} \\
\end{bmatrix} \\
\begin{bmatrix}
f_0 \\
f_1 \\
f_2 \\
f_3 \\
\end{bmatrix}
\]
FFT
FFT
FFT

Graphs showing the Fourier Transform (FFT) of a signal, with two separate axes and frequency ranges.
FFT
Harr Wavelet
Wavelet Representation

32 values
Daubichies Wavelet (6)
Daubichies Wavelet (6)
Daubichies Wavelet (6)
Daubichies Wavelet (6)
Daubichies Wavelet (6)
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Daubichies Wavelet (6)
Daubichies Wavelet (6)
Daubichies Wavelet (6)
Daubichies(6) vs. B-Spline(304)
Daubichies(6) vs. B-Spline(304)
\[
\bar{F}_k = \sum_{x=0}^{w} f(x) e^{-ikx} \rightarrow \\
\begin{bmatrix}
\bar{F}_0 & \bar{F}_1 & \bar{F}_2 & \bar{F}_3 \\
\end{bmatrix} = \\
\begin{bmatrix}
-e^{-ik_0 x_0} & e^{-ik_0 x_1} & e^{-ik_0 x_2} & e^{-ik_0 x_3} \\
e^{-ik_1 x_0} & e^{-ik_1 x_1} & e^{-ik_1 x_2} & e^{-ik_1 x_3} \\
e^{-ik_2 x_0} & e^{-ik_2 x_1} & e^{-ik_2 x_2} & e^{-ik_2 x_3} \\
e^{-ik_3 x_0} & e^{-ik_3 x_1} & e^{-ik_3 x_2} & e^{-ik_3 x_3} \\
\end{bmatrix} \begin{bmatrix}
f_0 \\
f_1 \\
f_2 \\
f_3 \\
\end{bmatrix}
\]
Orthogonality

\[ \int_{-\infty}^{\infty} \sin(k_1 x) \sin(k_2 x) \, dx = 0 \quad (k_1 \neq k_2) \]
Orthogonality

\[ \int_{-\infty}^{\infty} \sin(k_1 x) \sin(k_2 x) \, dx = 0 \quad (k_1 \neq k_2) \]

\[ \sum_{x=0}^{w} \sin(2\pi \frac{k_1 x}{w}) \sin(2\pi \frac{k_2 x}{w}) \, dx = 0 \quad (k \text{ integer, } k_1 \neq k_2) \]
Orthogonality
SVD / MSA?

- **Real Space** –
  - accurate position
  - no spectral information

- **FFT**
  - no positional information
  - accurate spectral information

- **Wavelets**
  - mix of power and position

- **MSA/SVD**
  - data-based basis
Computational Efficiency

Fourier (Matrix Method) : $O(n^2)$
Fourier (FFT) : $O(n \log n)$
Wavelet (std) : $O(n)$
Image Processing

- Filtration
- Deconvolution
- Transformation
- Registration
- Measures of Similarity
- Projection/Reconstruction (3D ->2D, 2D->3D)
- Segmentation

- Dimensionality?
Fourier Transform Theorems

if $f(x)$ real $\rightarrow \bar{F}(k) = \bar{F}^*(-k)$

Convolution: $f \ast g \rightarrow \bar{F}(k)\bar{G}(k)$

Correlation: $\int_{-\infty}^{\infty} g(x + a)h(a)\, da \rightarrow \bar{G}(k)\bar{H}^*(k)$

Translation: $f(x + x_0) \rightarrow \bar{F}(k) e^{ikx_0}$
Filtration -> Convolution

Continuous Real Space Convolution:
\[ f(x) * g(x) = \int_{-\infty}^{\infty} f(t) g(x-t) \, dt \]

Discrete Real Space Convolution:
\[ f_i * g_i = \sum_{t=-\infty}^{\infty} f_t g_{i-t} \]
Filtration -> Convolution

Continuous Real Space Convolution:
\[ f(x) * g(x) = \int_{-\infty}^{\infty} f(t) g(x - t) \, dt \]

Discrete Real Space Convolution:
\[ f_i * g_i = \sum_{t=-\infty}^{\infty} f_t g_{i-t} \]

![Diagram of convolution operation with bars and asterisk symbol]
Filtration -> Convolution

Continuous Real Space Convolution:
\[ f(x) * g(x) = \int_{-\infty}^{\infty} f(t) g(x-t) \, dt \]

Discrete Real Space Convolution:
\[ f_i * g_i = \sum_{t=-\infty}^{\infty} f_t g_{i-t} \]
Fourier Space and Images

\[ F(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) \, dx \]

\[ a + ib = \text{amp}(\cos \text{pha} + i \sin \text{pha}) \]
FFT Image demo

Real

FFT Amplitude
FFT Image demo

Real

FFT Phase
FFT Image demo

Real

Full FFT (Phase in Color)
FFT Image demo

Real

Full FFT
(Phase in Color)
FFT Image demo

Real

Full FFT
(Phase in Color)
FFT Image demo

Real

Full FFT
(Phase in Color)
Infinite/Continuous vs. Finite/Discrete Fourier Transform

- Finite -> Periodic

is really ->

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Gaussians

Gaussian or 'Normal' Distribution:

\[ f(x) = e^{-\left(\frac{x}{w}\right)^2} \]

\[ \int_{-\infty}^{\infty} e^{ikx} e^{-\left(\frac{x}{w}\right)^2} \, dx = w\sqrt{\pi} e^{-\left(\frac{k}{2}\right)^2} \]

Gaussians are everywhere
Fourier Convolution/Filtration

\[ \overline{F}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) \, dx \]

\[ h(x) = f(x) \ast g(x) = \int_{-\infty}^{\infty} f(t) g(x - t) \, dt \]

\[ \overline{H}(k) = \overline{F}(k) \overline{G}(k) \]
Test Image
Image Filtration
Gaussian Lowpass

$e^{-\frac{(\frac{x}{w})^2}{w}}$
Image Filtration
Sharp Lowpass

\[ x < x_c \rightarrow 1.0 \]
\[ \text{else} \rightarrow 0 \]
Image Filtration
Sharp Lowpass

\[ x < x_c \rightarrow 1.0 \]
\[ \text{else } \rightarrow 0 \]

\[ x_c = .125 \]
Image Filtration
Butterworth Lowpass

\[ \sqrt{1 + \left( \frac{X}{X_c} \right)^{2n}} \]

\[ x_c = .15 \]
\[ n = 8 \]
Image Filtration
Gaussian Highpass

\[ 1.0 - e^{-\left( \frac{x}{w} \right)^2} \]

\[ w = 0.125 \]
Deconvolution

$$e^{-\left(\frac{x}{w}\right)^2}$$
Deconvolution

\[ e^{\left(\frac{x}{w}\right)^2} \]
Deconvolution
Deconvolution

From Discrete valued image
How do we characterize the noise?
White Noise
Pink Noise
Optimal Filtration

\[ a_{x,y} + b_{x,y} = c_{x,y} \]
Optimal Filtration

\[ W(s_x, s_y) = F(s_x, s_y)C(s_x, s_y) \]

where

\[ \sum_{x,y} (c_{x,y} - a_{x,y})^2 \]

Answer is a Wiener Filter:

\[ F(s_x, s_y) = \frac{1}{1 + \frac{1}{\text{SNR}(s_x, s_y)}} \]
Wiener Filter

\[ F(s_x, s_y) = \frac{1}{1 + \frac{1}{\text{SNR}(s_x, s_y)}} \]
Wiener Filter
Wiener Filter
Wavelets in 2-D
Wavelets in 2-D

Harr
Wavelets in 2-D

Daub4
Wavelets in 2-D

Daub8
Wavelets in 2-D
Wavelets in 2-D

Daub8
Wavelets in 2-D

Strong threshold
-thr<x<thr  ->  0
Wavelets in 2-D

Less strong threshold (thr smaller)
-thr<x<thr  ->  0
Transformation

- Translation
- Rotation
- Scaling
- Skewning
- Non-linear transformations (tensors)

\{ Orthogonal, Similarity, Affine \}
Affine Transformations

\[
\begin{align*}
x' &= x m_{00} + y m_{01} + m_{02} \\
y' &= x m_{10} + y m_{11} + m_{12}
\end{align*}
\]

\[
\begin{bmatrix}
x' \\
y' \\
1
\end{bmatrix} =
\begin{bmatrix}
m_{00} & m_{01} & m_{02} \\
m_{10} & m_{11} & m_{12} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]

Combine transformations by matrix multiplication
Translation

\[
\begin{bmatrix}
    x' \\
    y' \\
    1
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & dx \\
    0 & 1 & dy \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix}
\]
Rotation

\[
\begin{bmatrix}
  x' \\
y' \\
1
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & -\sin \theta & 0 \\
  \sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]
Scaling

\[
\begin{bmatrix}
  x' \\
y' \\
1
\end{bmatrix}
= \begin{bmatrix}
  s_x & 0 & 0 \\
  0 & s_y & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
y \\
1
\end{bmatrix}
\]
Transformation -> Interpolation

- Bilinear
- Bicubic Splines
- Gaussian
- Fourier
- Arbitrary Kernel?

Bilinear:

\[ \nu_{xy} = xy \nu_{00} + x(1-y)\nu_{10} + (1-x)y\nu_{01} + (1-x)(1-y)\nu_{11} \]
Interpolation Effects
Measures of Similarity

- Correlation Coefficient
- Variance (transformed density)
- Variance (matched filter)
- Phase Residual
- Mutual Information
- etc.
Correlation

\[
\frac{1}{\sqrt{\sum_{x,y} a_{x,y}^2 \cdot \sum_{x,y} b_{x,y}^2}} \sum_{x,y} a_{x,y} b_{x,y}
\]

Normalized from -1.0 - 1.0
Registration

- Exhaustive search
- Minimization techniques
- Cross correlation
Cross Correlation

\[ c(x) = \text{Corr}(f(x), g(x)) = \int_{-\infty}^{\infty} f(x+t)g(t)\,dt \]

In Fourier Space:

\[ C(k) = F(k) G^*(k) \]
Model Bias

Base

Noisy

Align to

Base

Noisy

Align to
Model Bias

Base

Noisy (~10% contrast)

Align to

25  100  250  1000  2000
Model Bias

Base

Noisy (~10% contrast)

Align to
Model Bias

Base

Noisy (~10% contrast)

Align to

25 100 250 1000 2000
Model Bias

Base

Noisy

Align to

25 100 250 1000 2000
Model Bias

Base

Noisy

Align to

Iter x4

25
100
250
1000
2000
Model Bias

Base

Noisy

Align to

Iter x8

25 100 250 1000 2000
Model Bias

Base

Noisy (~10% contrast)

Align to

25

100

250

1000

2000
Model Bias

Base

Noisy

Align to

Iter x4

25  100  250  1000  2000
Science of Estimation

Caltech: Ph101, Order of Magnitude Physics

- How to make estimates.
- How to decide what physical effects are important in a given situation, or to understand how some system works the way it does.
- How to decide what terms in complicated equations can be omitted or simplified.
- How to figure out the general features of the solutions to equations, without actually solving the equations.

http://www.its.caltech.edu/~oom/homework.htm
Unit Analysis

acceleration -> m/s$^2$
speed -> m/s

At 9.8 m/s$^2$, how long does it take to achieve a speed of 60 mi/hr?

At 9.8 m/s$^2$, how long does it take to go 1000 m?
Unit Analysis

acceleration -> m/s²
speed -> m/s

At 9.8 m/s², how long does it take to achieve a speed of 60 mi/hr?
(1 mile = 1600 m, 1 hr = 3600 sec)

At 9.8 m/s², how long does it take to go 1000 m?
Unit Analysis

acceleration -> m/s^2
speed -> m/s

At 9.8 m/s^2, how long does it take to achieve a speed of 60 mi/hr?
(1 mile = 1600 m, 1 hr = 3600 sec)

At 9.8 m/s^2, how long does it take to go 1000 m?

actually \( t = \sqrt{\frac{2d}{a}} \)
There are about 100,000 pieces of space junk (exploded satellite fragments, trash left by astronauts, etc.) larger than 1 cm orbiting the earth at altitudes between 200 and 400 km. The orbits are more-or-less randomly oriented. At orbital velocities, a 1 cm marble can penetrate and disable Space Station Freedom. What is the probability that it will be disabled by a collision during the next decade?
The radius of the earth is $6 \times 10^8$ cm, a 200 km = $2 \times 10^7$ cm can be ignored. ie – orbital velocity at 200 km ~ orbital velocity at the surface.

$v = \sqrt{G \, M/r} \sim 8 \times 10^5$ cm/sec

Cross-section of space-station ~ size of house
~$100 \, m^2 = 10^6 \, cm^2$

Volume of 200 km orbital space ~ $4 \pi \, r^2 \times 2 \times 10^6 \, cm = 9 \times 10^{24} \, cm^3$ -> $1 \times 10^{-20} \, obj/cm^3$.

station covers $8 \times 10^5 \, cm/sec \times 10^6 \, cm^2 = 8 \times 10^{11} \, cm^3/sec$

decade = $3 \times 10^8 \, sec$ -> ~2 collisions/decade
If all the manure from all the animals consumed as food (cattle, chickens, swine, etc.) in the US during the past century were spread over the surface area of the US, how thick would the layer be?
The US Government recommended food pyramid recommends 140-240 g of meat, eggs, beans and nuts. So take an average of 200 g of meat per day. Over the past century, there has been an average of about 200 million people in the US.

\[200 \text{g/day-person} \times 2 \times 10^8 \text{people} \times 365 \text{days/yr} \times 100 \text{years} = 1.5 \times 10^{15} \text{g}\]

... Our inferred factor is in fact quite close to the empirically well-measured factor of 28 used in setting design and legal requirements for cattle farms. ie – $28 \times \text{food} = \text{feces}$

Poop barely floats so ~1 g/cm$^3$.
US area ~ 4000 km x 2000 km = $8 \times 10^{16}$ cm$^2$.

so, $(28 \times 1.5 \times 10^{15} \text{g} / (1 \text{g/cm}^3 \times 8 \times 10^{16} \text{cm}^2)) \sim .5 \text{cm}$
Biochemists Should Know

1 mole of anything weighs its atomic weight in grams
1 mole of gas \( \sim 22.4 \) liters = \( 2.24 \times 10^4 \) cm\(^3\)

water \( \sim 1 \) gm/cm\(^3\) = \( 1 \) gm/ml = \( 1 \) kg/l

0 \( ^{\circ} \)C = 273.16 K, room temp \( \sim 25 \) \( ^{\circ} \)C = 77 \( ^{\circ} \)F, \( \text{LN}_2 \) = 77 K, -196 \( ^{\circ} \)C

\( r_{\text{water}} \) \( \sim 1.9 \) Å = \( 1.9 \times 10^{-8} \) cm

Lipid membrane thickness \( \sim 50 \) Å

Typical Cell \( \sim 1\text{-}100 \) microns, prokaryotes \( \sim 1\text{-}10 \) microns

Protein mass \( \sim 117 \) Da \( \times \) \# residues

alpha-helix: 3.6 residues/turn, rise: 1.5 Å/residue, 2.3 Å radius

beta-sheet: 3.3 Å/residue, \( \sim 4.5 \) Å strand separation

Charged, negative (acidic) hydrophilic: Asp, Glu

Charged, positive (basic), hydrophilic: Lys, Arg, His

Polar, \( \sim \) hydrophilic: Ser, Thr, Cys, Tyr, Asn, Gln

Nonpolar, hydrophobic: Gly, Ala, Val, Leu, Ile, Met, Phe, Trp, Pro